The Sliding Window Discrete Fourier Transform

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The Sliding Window DFT

The Sliding Window Discrete Fourier Transform (SW-DFT) computes a time-frequency representation of a signal, and is useful for analyzing signals with local periodicities. Shortly described, the SW-DFT takes sequential DFTs of a signal multiplied by a rectangular sliding window function, where the window function is only nonzero for a short duration. Let $\mathbf{x} = [x_0, x_1, \dots, x_{N-1}]$ be a length N signal, then for length $n \leq N$ windows, the SW-DFT of $\mathbf{x}$ is:

$$a_{k,p} = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} x_{p-n+1+j} \omega_n^{-jk}$$
$$k = 0, 1, \dots, n-1$$
$$p = n-1, n, \dots, N-1 \quad (1)$$

where $\omega_n = \exp(\frac{i2\pi}{n}) = \cos(\frac{2\pi}{n}) + i\sin(\frac{2\pi}{n})$. The SW-DFT results in a $n \times P = N-n+1$ array, where n corresponds to frequencies and P corresponds to window position. A useful way to think about the SW-DFT is a multivariate time-series, where each time-series corresponds to a Fourier frequency $(\frac{2\pi k}{n})$.

The Fast Sliding Window DFT

Computing the SW-DFT by definition (Equation 1) takes Pn^2 operations. We can easily reduce this to $O(Pn\log(n))$ by replacing the DFT in each window position with a Fast Fourier Transform (FFT). We further reduce the computational complexity to $O(Pn)$ using the Fast SW-DFT algorithm (Richardson and Eddy (2017)). The Fast SW-DFT algorithm uses the tree data-structure of the Cooley-Tukey FFT to remove repeated calculations in overlapping windows.

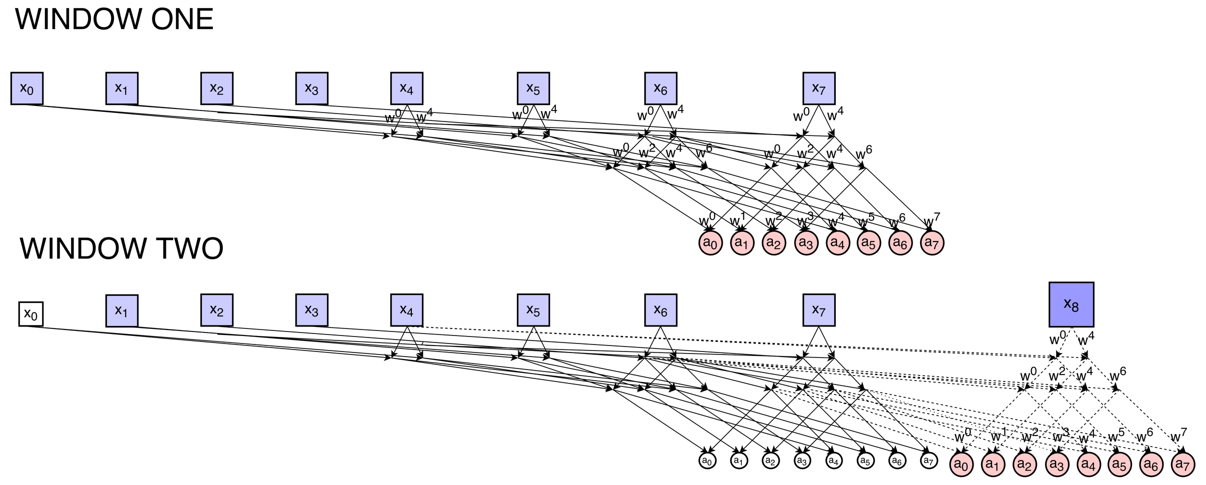


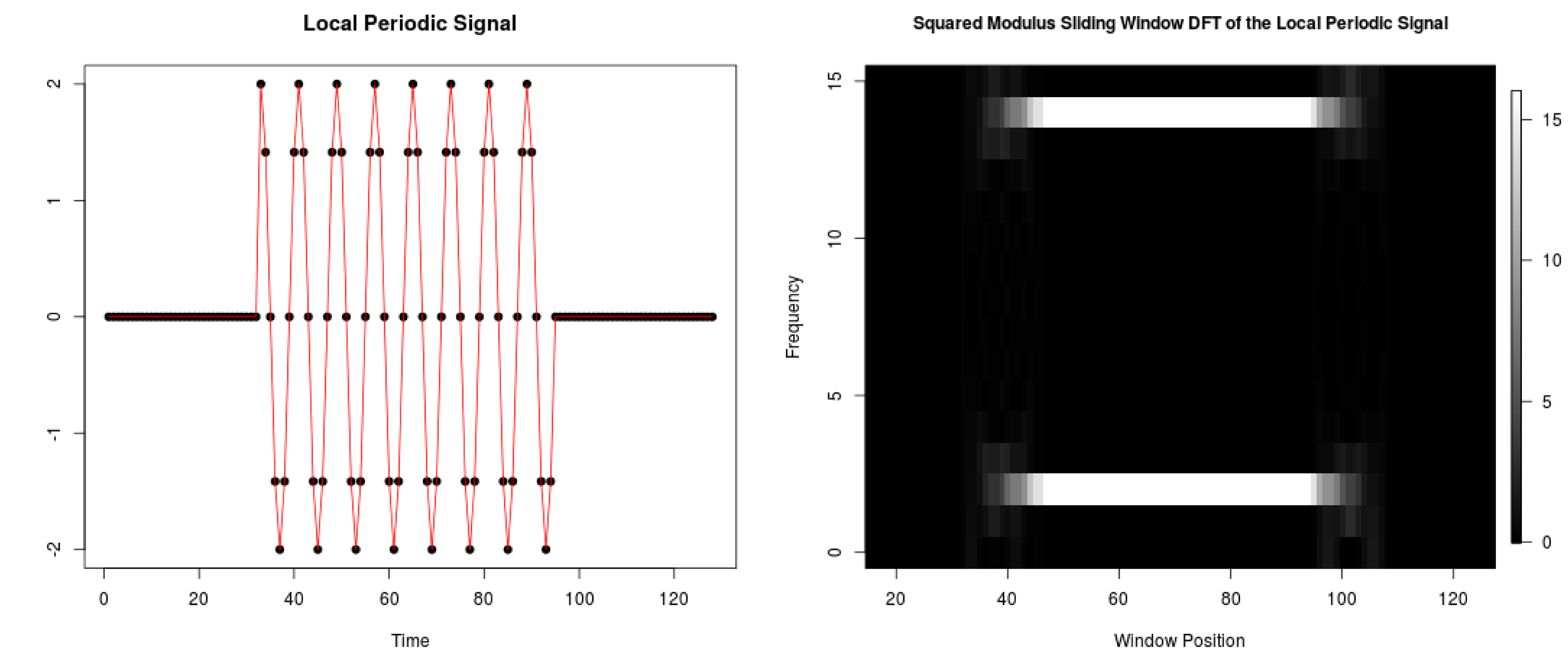
Figure: The Fast SW-DFT algorithm for two adjacent windows. Window one shows the intermediate FFT calculations for $[x_0, \dots, x_7]$, and window two shows the same calculations for $[x_1, \dots, x_8]$. The solid arrows in window two indicate calculations already made in window one, and the dashed arrows are new calculations. The new calculations in window two are exactly the size of a binary tree, which grows linearly with window size.

Sliding Window DFT for Local Periodic Signals

The Sliding Window DFT is a useful tool for analyzing data with local periodic signals (Richardson and Eddy (2018)). Since SW-DFT coefficients are complex-numbers, we analyze the squared modulus of these coefficients

$$|a_{k,p}|^2 = Re(a_{k,p})^2 + Im(a_{k,p})^2$$

Since the squared modulus SW-DFT coefficients are large when the window is on a periodic part of the signal:



Left: Length 128 signal $\mathbf{x}$, where $x_t = 2\cos(\frac{2\pi \cdot t \cdot 16}{128}) \cdot I_{[32,94]}(t)$. Right: Squared Modulus coefficients for the SW-DFT of $\mathbf{x}$.

Modeling a Local Signal

We use the following model for a local periodic signal:

$$y_t = A\cos(\frac{2\pi t F}{N} + \phi) \cdot I_{S,S+L-1}(t)$$

where $t = 0, \dots, N-1$. This model has 5 parameters:

1. S: Start of local signal. $S \in \{0, 1, \dots, N-2\}$.
2. L: Length of local signal. $L \in \{1, 2, \dots, N-S\}$
3. A: Amplitude. $A \in [0, \infty]$
4. F: Complete cycles in a length N signal. $F \in [0, \infty]$
5. ϕ: Phase, $\phi \in [0, 2\pi]$

The SW-DFT of our model for local signals is:

$$a_{k,p} = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} A\cos(\frac{2\pi(\hat{p}+j)F}{N} + \phi) \cdot I_{[S,S+L-1]}(\hat{p}+j)\omega_n^{-jk}$$

where $\hat{p} = p - n + 1$. This is easily extended to R local signals, using the linearity property of the Fourier transform.

Estimating a Local Signal

Let $\mathbf{x} = [x_0, x_1, \dots, x_{N-1}]$ be a length N signal. The SW-DFT of $\mathbf{x}$ is:

$$b_{k,p} = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} x_{p-n+1+j}\omega_n^{-jk} \quad (2)$$

We want the least-squares parameter estimates

$$\underset{S,L,A,F,\phi}{\arg\min} \sum_{k=0}^{n-1} \sum_{p=n-1}^{N-1} (b_{k,p} - a_{k,p})^2 \quad (3)$$

Using the following trigonometric identity:

$$\cos(x+y) = \cos(x)\cos(y) + \sin(x)\sin(y)$$

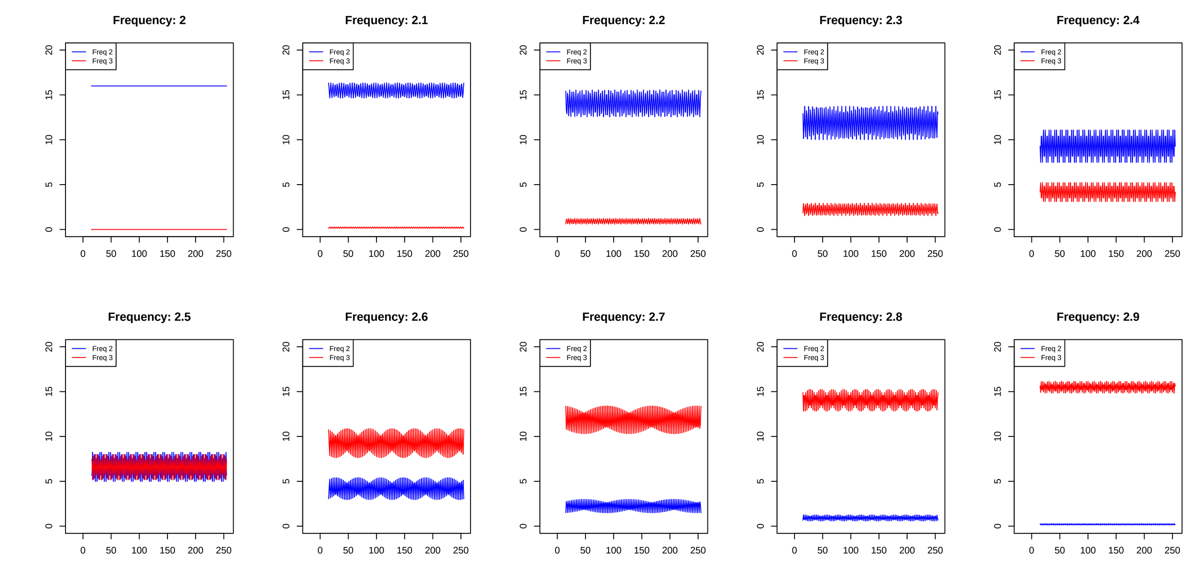
We can linearize the optimization over A and ϕ:

$$\underset{S,L,F}{\arg\min}(\underset{A,\phi}{\arg\min} \sum_{k=0}^{n-1} \sum_{p=n-1}^{N-1} (b_{k,p} - \beta_1 C_{1,k,p} - \beta_2 C_{2,k,p})^2)$$

Where $C_{1,k,p}$ and $C_{2,k,p}$ are complex-valued vectors (see Richardson and Eddy (2018) for more details).

Leakage

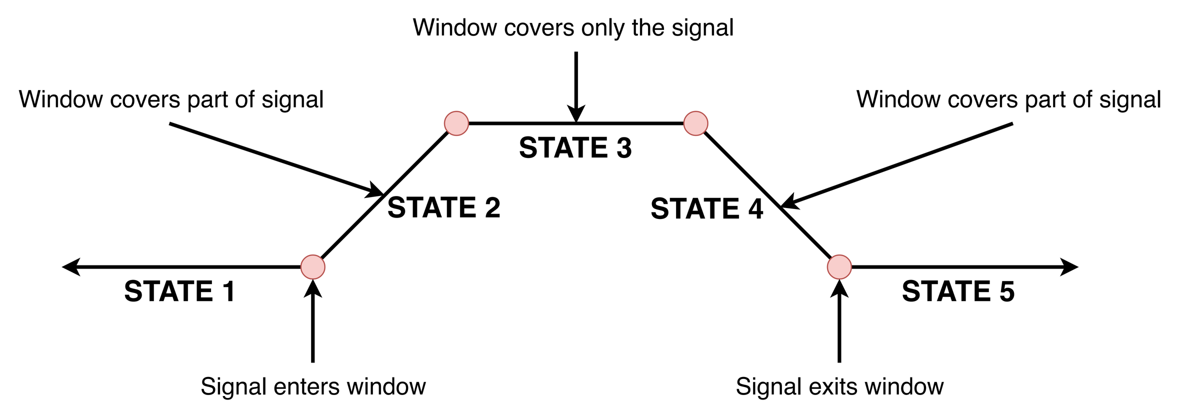
Leakage occurs when local signal frequency is not one of the Fourier frequencies (or, the number of complete cycles in a length n window is not an integer). In this case, the largest squared modulus coefficients occur at frequencies **closest** to the true frequency:



Squared Modulus SW-DFT coefficients for frequencies with 2 and 3 complete cycles, where true frequency ranges from $2 - 2.9$.

Ringing

Discontinuous signals lead to a phenomenon called **ringing**. For local signals, ringing occurs when the window is only on **part** of the local periodic signal, corresponding to states 2 and 4 in the figure below:



The canonical "trapezoid" shape of a squared modulus SW-DFT coefficient for a local periodic signal.

Conclusions

The SW-DFT is a useful tool for local periodic signals. This poster shows the linear-time Fast SW-DFT algorithm, a model for local periodic signals, a trick for estimating parameters of the model, and the "important-for-data-analysis" concepts: Leakage and Ringing.

References

Richardson, L. F. and Eddy, W. F. (2017). An algorithm for the 2d radix-2 sliding window fourier transform. *arXiv preprint arXiv:1707.08213*.

Richardson, L. F. and Eddy, W. F. (2018). The sliding window discrete fourier transform for time series. *In Preparation*.